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Normal forms and nonlinear symmetries

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Abstract. We give some general theorems, and extensions of previous results, concerning the problem of transforming an algebra of vector fields into Poincaré normal form. By means of a unifying algebraic language, we show the possibility of obtaining either a 'parallel' or 'joint' normal form of the vector fields in a definite way, which simplifies the construction of normal forms, providing a precise restriction on their structure. The application to finite-dimensional dynamical systems and their Lie point symmetries is also discussed.

1. Introduction and notations

The problem of transforming a vector field (VF) (or an algebra of VFs) into normal form (NF) (in the sense of Poincaré–Dulac–Birkhoff) is an old and important topic [1–10], not only for its algebraic aspects, but also for its applications in the theory of dynamical systems (DS), especially in connection with symmetry properties [1–12].

Quite different approaches and points of view ('algebraic', 'analytical' or 'dynamical') for this problem can be found in the literature and it is not uncommon for the differences in the language to make it difficult to compare (even apparently unrelated but strongly connected) results.

In this paper, we propose some general results, and extensions of previous statements, in an (essentially) self-contained presentation, using a geometrical approach similar to that in [12] with an abstract and 'unifying' algebraic language, but avoiding, as far as possible, any technicality (section 2). The applications to DS and their symmetries (Lie point symmetries) are discussed in section 3.

Let us recall some basic definitions and fix our notation. Let $u \in M \subseteq R^n$, where M is a smooth neighbourhood of the origin in R^n, and consider the space $\mathcal{V}$ of analytical VFs $\varphi : M \to TM$ in R^n: they are in one-to-one correspondence with the elements of the space V of analytical functions $f : M \to R^n$; in component expansion, we shall write (here, and in the following, summed over repeated indices, unless otherwise stated)

$$\varphi \equiv f(u)\partial_u \equiv f_i(u)\frac{\partial}{\partial u_i} \qquad (i = 1, \ldots, n). \tag{1.1}$$

We assume that $u = 0$ is an isolated fixed point for $f(u)$; $f(u)$ will also be written as a series expansion in the form

$$f \equiv Au + \tilde{f} \equiv \sum_{j=1}^{\infty} f_{(j)} \tag{1.2}$$

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where $Au \equiv f_{(1)}$ is the linear part of f, $\tilde{f}$ is the nonlinear part and $f_{(j)} \in V_{(j)}$ is the subspace of the homogeneous polynomial functions in V of degree j.

Given two VFs, $\varphi = f\partial_u$ and $\psi = g\partial_u$ in $\mathcal{V}$, the notion of Lie commutator $[\varphi, \psi]$ in $\mathcal{V}$ induces a Lie–Poisson bracket $\{f, g\}$ in V:

$$[\varphi, \psi] = \{f, g\}\partial_u \qquad \{f, g\}_k = f_i\partial_i g_k - g_i\partial_i f_k \qquad \left(\partial_i \equiv \frac{\partial}{\partial u_i}\right). \quad (1.3)$$

Also, a notion of scalar product can be introduced in each subspace $V_{(j)}$ [9, 12].

Given an $n \times n$ matrix A, we denote by $\mathcal{A} : V \to V$ the homological operator associated to A (which is also the Lie derivative $\mathcal{L}_A$ along the VF $Au\partial_u$)

$$\mathcal{A}(h_k) = (Au)_i\partial_i h_k - (Ah)_k \quad (1.4)$$

where $h = h(u) \in V$. According to the classical Poincaré–Dulac–Birkhoff definition [1], a (nonlinear) term $h(u)$ is said to be *resonant with* A if (see also [9])

$$\mathcal{A}^+(h) \equiv \{A^+u, h\} = 0 \quad (1.5)$$

and a VF, $\varphi = (Au + \tilde{f})\partial_u$, is said to be in normal form (NF) if all nonlinear terms are resonant with A, i.e. $\tilde{f} \in \operatorname{Ker}\mathcal{A}^+$. If A is diagonal, with eigenvalues $\alpha_1, \ldots, \alpha_n$, a monomial $h_k(u) = u_1^{m_1} \cdot \ldots \cdot u_n^{m_n}$ of degree j (with m_i integer numbers such that $\sum_i m_i = j$, $m_i \geqslant 0$) is resonant if $m_i\alpha_i = \alpha_k$, which is the usual 'resonance condition' for the eigenvalues [1]. As is well known, the relevance of the above definitions is essentially due to the fact that, given a VF φ, all non-resonant terms can be removed by means of a coordinate transformation. As usual, in NF theory, these transformations are expressed by means of *formal* series, i.e. no assumption is made on their convergence (cf [1]).

Notice that for both operators $\mathcal{A}$ and $\mathcal{A}^+$ one has, for each j,

$$\mathcal{A} : V_{(j)} \to V_{(j)} \qquad \text{and} \qquad \mathcal{A}^+ : V_{(j)} \to V_{(j)}. \quad (1.6)$$

If $\mathcal{A} = \mathcal{L}_A$ and $\mathcal{B} = \mathcal{L}_B$ are the homological operators associated to two $n \times n$ matrices A and B, the operator $\mathcal{AB} - \mathcal{BA}$ is just the homological operator $\mathcal{C} = \mathcal{L}_C$ associated to the matrix commutator $C = [A, B]$. In particular, the three statements $\mathcal{AB} = \mathcal{BA}$, $[A, B] = 0$ and $\{Au, Bu\} = 0$ are equivalent.

Finally, if A is any $n \times n$ matrix, we shall denote its (unique) decomposition into commuting semisimple (diagonalizable) and nilpotent parts by

$$A = A_s + A_n. \quad (1.7)$$

2. The algebraic approach

The main results will be obtained as a consequence of a series of simple lemmas, which can be of some independent interest: even if some of these are not new, it is convenient to give a complete list of all of them, together with a sketch of their proofs.

Lemma 1. If $[A, B] = 0$ then $[A_s, B] = [A, B_s] = 0$.

Proof. This easily follows from writing the matrix A (or resp. B) in its Jordan form and then imposing the commutation properties in the generalized eigenspaces. □

Lemma 2. Given the matrix A, the set $\operatorname{Ker}\mathcal{A}^+$ of terms $h(u)$ resonant with A is given by $K(\kappa(u))u$, where K is the most general matrix such that $[K, A^+] = 0$ and its entries K_{ij} are functions of the time-independent analytical constants of motion $\kappa = \kappa(u)$ of the linear system $\dot{u} = A^+u$ [9, 12].

Proof. Writing the equation $\mathcal{A}^+(h) = 0$ explicitly as a system of first-order partial differential equations

$$(A^+u)_i \frac{\partial h_k}{\partial u_i} = (A^+h)_k \tag{2.1}$$

one obtains a system 'with the same principal part' [13], with characteristic equations

$$\frac{\mathrm{d}u_1}{(A^+u)_1} = \cdots = \frac{\mathrm{d}u_n}{(A^+u)_n} = \frac{\mathrm{d}h_1}{(A^+h)_1} = \cdots = \frac{\mathrm{d}h_n}{(A^+h)_n}. \tag{2.1'}$$

Applying standard procedures [13, 9, 12], one obtains that $h = Ku$, where K is a matrix commuting with A^+ and depends on the constants of integration of the subset of equations involving the first n terms $\mathrm{d}u_j/(A^+u)_j$ in (2.1′). These are just the constants of motion of the problem $\dot{u} = A^+u$. □

Lemma 3. $\operatorname{Ker}\mathcal{A} \subset \operatorname{Ker}\mathcal{A}_s$; $\operatorname{Ker}\mathcal{A}^+ \subset \operatorname{Ker}\mathcal{A}_s$, where $\mathcal{A}_s$ is the homological operator associated to the semisimple part A_s of A.

Proof. According to lemma 1, if K commutes with A^+, then it also commutes with A_s $(= A_s^+)$. On the other hand, the solutions of the linear systems $\dot{u} = A_s u$ and $\dot{u} = A^+u$ are combinations of terms $\mathrm{e}^{\alpha_j t}$ and $t^k \mathrm{e}^{\alpha_j t}$, respectively. Therefore, the constants of motion which can be constructed for the second system, with the property of being time-independent and expressed in analytical form, are certainly also constants of motion of the first system (but the converse is not true). The statement of lemma 3 then follows from lemma 2. □

Lemma 4. If $\varphi = (Au + \tilde{f})\partial_u$ and $\psi = (Bu + \tilde{g})\partial_u$ form a two-dimensional algebra, then it is possible to perform a (formal) coordinate transformation which takes the nonlinear terms $\tilde{f}$ into normal form with respect to A, and $\tilde{g}$ with respect to B ('*parallel* normal form').

Proof. Up to a linear transformation, any two-dimensional algebra satisfies the commutation rule

$$[\varphi, \psi] = c\psi \tag{2.2}$$

where c is any constant (including $c = 0$). First of all, we can always put $\tilde{g}$ into NF with respect to B, so, let us assume (without changing notation)

$$\tilde{g} \in \operatorname{Ker}\mathcal{B}^+$$

where $\mathcal{B}^+$ is the homological operator associated to B^+. Now, from (2.2), $[A, B] = cB$, or $[A^+, B^+] = -\bar{c}B^+$, which implies, in terms of the homological operators,

$$\mathcal{B}^+(\mathcal{A}^+(\tilde{g})) = \mathcal{A}^+(\mathcal{B}^+(\tilde{g})) - \bar{c}\mathcal{B}^+(\tilde{g}) = 0$$

i.e. $\mathcal{A}^+(\tilde{g}) \in \operatorname{Ker}\mathcal{B}^+$ or $\mathcal{A}^+ : \operatorname{Ker}\mathcal{B}^+ \to \operatorname{Ker}\mathcal{B}^+$. This ensures the possibility of performing another transformation, leaving the space $\operatorname{Ker}\mathcal{B}^+$ of the terms resonant with B invariant in such a way as to change the terms $\tilde{f}$ into NF with respect to A. Then, we can choose coordinates in such a way that:

$$\tilde{f} \in \operatorname{Ker}\mathcal{A}^+ \qquad \text{and} \qquad \tilde{g} \in \operatorname{Ker}\mathcal{B}^+.$$ □

We can now state the first main result.

Theorem 1. Let $\varphi = f(u)\partial_u = (Au + \tilde{f})\partial_u$, $\psi = g(u)\partial_u = (Bu + \tilde{g})\partial_u$ satisfy

$$[\varphi, \psi] = 0 \tag{2.3}$$

then, by means of a formal coordinate transformation, $\tilde{f}, \tilde{g}$ can be taken into a '*joint* normal form' (JNF) of this type:

$$\tilde{f} \in \operatorname{Ker} \mathcal{A}^+ \cap \operatorname{Ker} \mathcal{B}_s \qquad \text{and} \qquad \tilde{g} \in \operatorname{Ker} \mathcal{A}_s \cap \operatorname{Ker} \mathcal{B}^+. \tag{2.4}$$

Proof. From lemmas 3 and 4, we get

$$\tilde{f} \in \operatorname{Ker} \mathcal{A}^+ \subset \operatorname{Ker} \mathcal{A}_s \qquad \text{and} \qquad \tilde{g} \in \operatorname{Ker} \mathcal{B}^+ \subset \operatorname{Ker} \mathcal{B}_s. \tag{2.5}$$

Let us now write (2.3) step by step, with

$$f(u) = Au + \sum_{j=2}^{\infty} f_{(j)} \qquad \text{and} \qquad g(u) = Bu + \sum_{j=2}^{\infty} g_{(j)}. \tag{2.6}$$

We have first

$$[A, B] = 0 \tag{2.7}$$

and

$$\{Au, g_{(2)}\} - \{Bu, f_{(2)}\} = 0 \qquad \text{or} \qquad \mathcal{A}(g_{(2)}) = \mathcal{B}(f_{(2)}) \tag{2.8}$$

whereas, for $k > 2$,

$$\{Au, g_{(k)}\} - \{Bu, f_{(k)}\} = \sum_{j=2}^{k-1} \{f_{(j)}, g_{(k-j+1)}\}. \tag{2.9}$$

Applying operator $\mathcal{A}_s$ to (2.8), and using lemma 1, we obtain, thanks to (2.5),

$$\mathcal{A}_s(\mathcal{A}(g_{(2)})) = \mathcal{B}(\mathcal{A}_s(f_{(2)})) = 0$$

and also $\mathcal{A}_s^2(g_{(2)}) = 0$, which implies

$$\mathcal{A}_s(g_{(2)}) = 0.$$

In fact, since A_s is a diagonalizable matrix, we can choose coordinates such that $\operatorname{Ker} \mathcal{A}_s$ is the orthogonal complement to $\operatorname{Ran} \mathcal{A}_s$ in the space $V_{(2)}$. Repeating the argument for the operator $\mathcal{B}_s$ applied to (2.8), we get similarly

$$\mathcal{B}_s(f_{(2)}) = 0.$$

An immediate application of the Jacobi identity shows that if

$$f_{(j)},\ g_{(i)} \in \operatorname{Ker} \mathcal{A}_s \cap \operatorname{Ker} \mathcal{B}_s \qquad \forall\, i, j = 2, \ldots, k-1$$

then the same is true for

$$\{f_{(j)}, g_{(i)}\} \qquad \text{and} \qquad \sum_{j=2}^{k-1} \{f_{(j)}, g_{(k-j+1)}\}. \tag{2.10}$$

This allows us to proceed inductively: applying the operators $\mathcal{A}_s$ and $\mathcal{B}_s$ to (2.10), we can conclude, for all j, that $f_{(j)} \in \operatorname{Ker} \mathcal{B}_s$ and $g_{(j)} \in \operatorname{Ker} \mathcal{A}_s$, which, together with (2.5), gives the result. □

The possibility of extending the above results (namely, lemma 4 and theorem 1) to algebras of dimension $d > 2$ is clearly related to the specific commutation properties of the algebra. We consider here some special cases.

Theorem 2. Let us consider a d-dimensional algebra $\mathcal{G}$ of VFs spanned by $\varphi_a = f_a \partial_u = (A_a u + \tilde{f}_a)\partial_u$ $(a = 1, \dots, d)$. Then:

(i) If the algebra $\mathcal{G}$ is *solvable*, then all the nonlinear terms $\tilde{f}_a$ can be put in parallel NF, namely

$$\tilde{f}_a \in \operatorname{Ker} \mathcal{A}_a^+ \qquad \text{for each } a = 1, \dots, d. \tag{2.11}$$

(ii) If the algebra $\mathcal{G}$ is *nilpotent* (in particular: abelian), then one can put all $\tilde{f}_a$ into a JNF precisely (with obvious notations)

$$\tilde{f}_a \in \left(\bigcap_{b \neq a} \operatorname{Ker} \mathcal{A}_{b,s} \right) \cap \operatorname{Ker} \mathcal{A}_a^+ \qquad \text{for each } a = 1, \dots, d. \tag{2.12}$$

(iii) In any solvable (resp.: nilpotent, or, in particular, abelian) *subalgebra* of a generic algebra $\mathcal{G}$, all nonlinear terms can be put in parallel NF as in (2.11) (resp.: in JNF as in (2.12)).

Proof. If the algebra is solvable, let us consider the sequence of commutators terminating in 0

$$[\varphi, \varphi] = \varphi^{(1)}, [\varphi^{(1)}, \varphi^{(1)}] = \varphi^{(2)}, \dots, [\varphi^{(m)}, \varphi^{(m)}] = 0 \tag{2.13}$$

where $[\varphi, \varphi] = \varphi^{(1)}$ stands for the subalgebra of all commutators $[\varphi_a, \varphi_b]$, etc. In the ideal $\mathcal{G}^{(m)}$, spanned by $\varphi^{(m)}$, all nonlinear terms of the VFs can be taken in NF (or even in JNF if $\dim \mathcal{G}^{(m)} > 1$: indeed, in this case the subalgebra is abelian and theorem 1 can be directly applied). Using $[\varphi^{(m-1)}, \varphi^{(m-1)}] = \varphi^{(m)}$ and $\mathcal{G}^{(m-1)} \supseteq \mathcal{G}^{(m)}$, we can repeat the argument of lemma 4 to also show that in $\mathcal{G}^{(m-1)}$, parallel NFs can be obtained, and so on. If, now, the algebra $\mathcal{G}$ is nilpotent, let us consider the sequence of commutators terminating in 0

$$[\varphi, \varphi] = \varphi^{[1]} (= \varphi^{(1)}), [\varphi, \varphi^{[1]}] = \varphi^{[2]}, \dots, [\varphi, \varphi^{[m]}] = 0. \tag{2.14}$$

The first part of this theorem ensures (since nilpotency implies solvability) that all $\tilde{f}_a$ can be taken in their respective NF: $\tilde{f}_a \in \operatorname{Ker} \mathcal{A}_a^+$. On the other hand, the last commutator in (2.14) shows that all fields in the abelian ideal $\mathcal{G}^{[m]}$ spanned by $\varphi^{[m]}$ commute with all the $\varphi_a \in \mathcal{G}$. Then, the procedure followed in the proof of theorem 1 can be repeated for each $\varphi_a \in \mathcal{G}$, using the last commutator in (2.14) to obtain (2.12). Statement (iii) is an immediate consequence. □

Remark 1. The result in theorem 1 and its extension in theorem 2(ii) are generalizations of theorem 2.2† of [7], which gives in fact $\tilde{f}_a \in \bigcap_b \operatorname{Ker} \mathcal{A}_{b,s}$. Notice that condition $\tilde{f} \in \operatorname{Ker} \mathcal{A}^+$ is actually a rather stronger restriction for $\tilde{f}$ than $\tilde{f} \in \operatorname{Ker} \mathcal{A}_s$, the space $\operatorname{Ker} \mathcal{A}^+$ being, in general, considerably smaller than the space $\operatorname{Ker} \mathcal{A}_s$, as simple examples can easily show. Notice also that, in general, it is not possible to extend the result in theorem 1 (and in 2(ii) as well) to also have, for example, $\tilde{f} \in \operatorname{Ker} \mathcal{B}$ or $\tilde{f} \in \operatorname{Ker} \mathcal{B}^+$. The case of solvable algebras is quite different: e.g. if $[\varphi, \psi] = \psi$, then $[A, B] = B$, but this implies $B_s = 0$ and therefore one gets, in this case, $\mathcal{B}_s = 0$.

† Any two-dimensional nilpotent algebra is in fact abelian. Unfortunately, reference [7] came to our knowledge only after our paper [12]—which contains results already obtained in [7], although by different methods—was published.

Remark 2. In NF theory, it is usual to give special attention to the case of VFs with *normal* linear parts, i.e. $[A, A^+] = 0$ [12]. Here, we will not consider this restriction: the general results given here can of course be specialized to this case (obtaining, among others, some of the results given in [12]). For instance, equation (2.12) becomes immediately, with this restriction,

$$\tilde{f}_a \in \bigcap_b \mathrm{Ker}\,\mathcal{A}_b. \tag{2.12'}$$

3. Applications to DS and their symmetry properties

Let us now apply the above algebraic results to the case of (finite-dimensional) DS. With $u = u(t) \in M \subseteq R^n$, let

$$\dot{u} = f(u) = Au + \tilde{f}(u) \tag{3.1}$$

be a DS, where $\dot{u} = \mathrm{d}u/\mathrm{d}t$ and $f(0) = 0$.

Denoting by $\varphi \equiv f\partial_u$ the VF expressing the dynamical flow of this DS, any VF $\psi = g\partial_u$, such that

$$[\varphi, \psi] = 0 \tag{3.2}$$

is the generator of a Lie point time-independent (LPTI) symmetry of this DS [14–16] (see also [11, 12, 17–21] and references therein). Therefore, according to theorem 1, one can choose coordinates in such a way that both VFs φ and ψ are in JNF (2.4). In concrete cases, once the DS is given, a typical problem is that of finding its LPTI symmetries: then, the set of equations (2.7)–(2.9) may be used, in practice, in order to construct, recursively, step by step, the symmetry field, and the JNF condition (2.4) determines the nonlinear terms which may be removed, both in the DS and in the VF describing the symmetry.

Remark 3. Clearly, it is not granted, in general, that, as for the NF transformations, the LPTI symmetries of a DS can be written as a series expansion (for a very recent and important result in this direction, see [22]; examples of 'singular' LPTI symmetries can be found, e.g., in [18, 19]). Let us notice, however, that the method of proceeding step by step may be concretely useful in constructing 'approximate' symmetries, i.e. 'up to the a given (finite) order' [17] (see also [23]). Some sufficient conditions ensuring the existence of (polynomial) LPTI symmetries, and some explicit examples, can be found in [11, 12, 18–21]. A (linear) symmetry which is always present (unless A_s, the semisimple part of A, is $= 0$) is given by the following proposition.

Proposition 1. Any DS (3.1) which is in NF, i.e. with $\tilde{f} \in \mathrm{Ker}\,\mathcal{A}^+$, admits the linear symmetry generated by

$$\sigma = A_s u \partial_u. \tag{3.3}$$

If A_s is diagonalized, with real eigenvalues α_i ($i = 1, \ldots, n$), this symmetry generates the scaling $u_i \to u_i \exp(\varepsilon\alpha_i)$ ($\varepsilon \in R$).

Proof. Symmetry condition (3.2) is certainly satisfied by this σ (3.3), indeed

$$\{Au + \tilde{f}, A_s u\} = \{\tilde{f}, A_s u\} = 0$$

as a consequence of the resonance assumption and lemma 3. □

This generalizes proposition 4 of [11], where a diagonalizable A was assumed, and another result contained in [9] which—in the present language—may be stated as follows. If the DS (3.1) is in NF, then $(A^+ u)\partial_u$ is a symmetry for the *nonlinear* part of the DS $\dot{u} = \tilde{f}$ (but not necessarily for the *full* DS $\dot{u} = f$). Another property of LPTI symmetries and NFs, which is a direct consequence of (2.4), is given by the following proposition.

Proposition 2. If the DS (3.1) admits a LPTI symmetry $\psi = g\partial_u = (Bu + \tilde{g})\partial_u$ and the fields φ, ψ are in JNF, then ψ is also a symmetry for the linear semisimple part of the DS, i.e. for $\dot{u} = A_s u$ (but the converse is not true: i.e. symmetries of this linear system are not necessarily symmetries for the full DS) and the linear semisimple part $B_s u \partial_u$ provides another symmetry (if $B_s \neq 0$) for the DS:

$$\{A_s u, g\} = 0 \qquad \text{and} \qquad \{B_s u, f\} = 0. \tag{3.4}$$

Clearly, the set $\{\psi_1, \ldots, \psi_r\}$ of the LPTI symmetries of a DS spans a Lie algebra† $\mathcal{G}$: it always contains the VF φ giving the dynamical flow. This algebra may be abelian (as in proposition 3 below) or not. An interesting example of non-abelian symmetry algebra can be constructed starting from a four-dimensional problem which has the 'quaternionic structure' [24]: the generators of its symmetry span the Lie algebra of the group $SU(2)$ (in real form).

The possibility of taking in parallel, or JNF, the VFs $\{\varphi, \psi_1, \ldots, \psi_r\}$ in this algebra depends on the properties of the algebra itself, according to theorem 2. In any case, however, being $[\varphi, \psi_a] = 0$ for all $a = 1, \ldots, r$, by the definition of symmetry, the JNF is possible for φ and at least *one* of the ψ_a, or better for all the ψ_a which span an abelian subalgebra $\mathcal{H} \subseteq \mathcal{G}$.

Let us consider finally the special case of *linearizable* DSs, i.e. DSs for which all terms are non-resonant and, therefore, can be removed by a (formal) coordinate transformation.

Remark 4. According to the definition, a DS (3.1) is linearizable if $\operatorname{Ker} \mathcal{A}^+ = \{0\}$. However, a simple consequence of JNF also indicates that a condition ensuring the linearizability of a DS is that it admits an LPTI symmetry $\psi = (Bu + \tilde{g})\partial_u$ such that $\operatorname{Ker} \mathcal{A}^+ \cap \operatorname{Ker} \mathcal{B}_s = \{0\}$ and, in general, the same is true if $(\bigcap_h \operatorname{Ker} \mathcal{B}_{h,s}) \cap \operatorname{Ker} \mathcal{A}^+ = \{0\}$, where $\psi_h = (B_h u + \tilde{g}_h)\partial_u \in \mathcal{H}$ and $\mathcal{H}$ is an abelian subalgebra contained in the algebra $\mathcal{G}$ of the admitted symmetries, as mentioned previously.

We also have:

† Actually, multiplying an LPTI symmetry by any constant of motion of the DS gives another symmetry, so one should more correctly speak of a (finite-dimensional) *module* (rather than of an infinite-dimensional algebra) of symmetries [20]. Here, we are interested in the algebraic structure, and, therefore, we are considering only 'independent' (i.e. pointwise linearly independent) VFs generating symmetries.

Proposition 3. If a DS can be linearized, then it admits n independent *commuting* symmetries, which can be simultaneously taken into *linear* form by a coordinate transformation. If, in particular, the system has a diagonalizable A with real eigenvalues, then—once it is linearized and A is diagonal—the dilations $\zeta_i = u_i\partial_i$ (no sum over i) along each direction i are n linear commuting symmetries for the system. Conversely, if there is a coordinate system where the DS admits n independent linear commuting symmetries $\sigma_i = B_i u\partial_u$ such that all B_i are semisimple, then the DS can be linearized.

Proof. In the coordinates where the DS is linear, it is easy to construct n linear commuting symmetries $B_i u\partial_u$ by simply choosing n independent matrices B_i commuting among themselves and with A (if the matrices $I, A, A^2, \ldots, A^{n-1}$ are linearly independent, they immediately provide the matrices B_i; even if this is not the case, the existence of n matrices B_i with the required properties is easily verified if A is put in Jordan form). The existence of n independent scalings ζ_i in the case of diagonal A is obvious. Conversely, given n semisimple commuting matrices B_i, they can be simultaneously diagonalized: $B_i \to \mathrm{diag}(\beta_1^{(i)}, \ldots, \beta_n^{(i)})$; now, with respect to the basis spanned by the n (independent) vectors $\beta^{(i)} \equiv (\beta_1^{(i)}, \ldots, \beta_n^{(i)})$, the symmetries $\sigma_i = B_i u\partial_u$ become $\sigma_i \to \zeta_i = u_i\partial_i$ (no sum over i), i.e. the independent dilations along the directions $\beta^{(i)}$. A DS admitting such n symmetries is necessarily linear.

4. Examples and concluding remarks

We shall first illustrate the basic preliminary results of lemmas 2 and 3 by means of a simple example. Let us consider the VF $\varphi = f\partial_u$ with $n = 3$

$$f = Au + h(u) \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{4.1}$$

where A is non-diagonalizable. Considering the two linear problems $\dot{u} = A_s u$ and $\dot{u} = A^+ u$, and putting $u \equiv (x, y, z)$, one directly obtains that the quantity $\kappa = x^2 z$ is a constant of motion for both problems, whereas $y^2 z$ is another constant of motion for the first problem but not for the second: in fact, any other independent constant of motion for the second problem would be expressed by means of a non-analytic form, e.g. $xyz - x^2 z \log x$. One then immediately deduces that the terms resonant with A, i.e. $h \in \mathrm{Ker}\,\mathcal{A}^+$, are given by

$$h(u) = \begin{cases} a(y^2z)x + b(y^2z)y \\ a(y^2z)y \\ c(y^2z)z \end{cases} \tag{4.2}$$

where a, b, c are (smooth) arbitrary functions of $y^2 z$. Instead, the terms resonant with A_s, $h' \in \mathrm{Ker}\,\mathcal{A}_s$, are given by

$$h'(u) = \begin{cases} a'(x^2z, y^2z)x + a''(x^2z, y^2z)y \\ b'(x^2z, y^2z)x + b''(x^2z, y^2z)y \\ c'(x^2z, y^2z)z \end{cases}$$

showing that $\mathrm{Ker}\,\mathcal{A}^+ \subset \mathrm{Ker}\,\mathcal{A}_s$. Notice also that the linear operator

$$\sigma = x\partial_x + y\partial_y - 2z\partial_z \tag{4.3}$$

generates a symmetry for any DS, which is described by the above VF $\varphi = f\partial_u$ (4.1), once it is put in NF (i.e. with $h(u)$ in the form (4.2)): this symmetry is just the scaling (leaving, in particular, the quantity y^2z invariant, cf (4.2)) in agreement with proposition 1 of section 3.

An example of VF which is in NF and admits a nonlinear (nonlinearizable) symmetry is given by

$$f(u) = \begin{cases} x + x^3z \\ y + y^3z \\ -2z \end{cases}$$

which admits, in addition to the scaling (4.3), the symmetry operator

$$\sigma = x^3z\partial_x.$$

As already remarked, the main result of this paper is essentially of algebraic nature. In addition to the examples mentioned briefly in the previous section, in the context of the applications to the DS theory, let us also propose the following further example. Consider the DS (with $n = 3$, $u \equiv (x, y, z)$, $r^2 = x^2 + y^2$)

$$\begin{aligned} \dot{x} &= -y + x(r^2 - ze^{-y}) \\ \dot{y} &= x + y(r^2 - ze^{-y}) \\ \dot{z} &= -z + r^2e^y + yz(r^2 - ze^{-y}) + xz. \end{aligned} \tag{4.4}$$

It is easy to verify (cf [20], where a more general class of examples are presented) that it admits the following nonlinear symmetry:

$$\sigma = y\partial_x - x\partial_y - xz\partial_z. \tag{4.5}$$

Both the DS (4.4) and the symmetry (4.5) contain non-resonant terms, but one can easily verify that, according to the above results, once reduced to NF, the DS is transformed into a DS exhibiting rotational symmetry $\sigma = y\partial_x - x\partial_y$. It can be interesting to compare this result with the one obtained under another coordinate transformation which is usually considered when dealing with DS, namely, the transformation to the coordinates of centre manifold. In this example, the (unique) centre manifold is given by

$$z = r^2e^y.$$

Putting $\zeta = z - r^2e^y$, the DS (4.4) is transformed into

$$\begin{aligned} \dot{x} &= -y - x\zeta e^{-y} \\ \dot{y} &= x - y\zeta e^{-y} \\ \dot{\zeta} &= -\zeta - y\zeta^2e^{-y} + x\zeta + 2r^2\zeta \end{aligned} \tag{4.6}$$

and one can see that this system admits the rotation symmetry *only on the centre manifold* $\zeta = 0$.

The relationships between Lie symmetries, NF transformations, centre-manifold reductions and other techniques in the theory of DS, including bifurcation theory and Hamiltonian problems, are the subject of a number of works: we refer the reader, for example, to [3–5, 8–10, 22] (see also [19–21, 25] and references therein), where a detailed discussion can be found.

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